

PART V. APPENDIX: OPTIMAL MUTATION PROBABILITY

Suppose we have a fitness function that is a function of n binary variables and is a monotone function of the Hamming distance of its argument vector from the solution vector: $f(x) = g(\|x - x^{(0)}\|)$, where g is a monotone function defined on the integers $0, \dots, n$, $x^{(0)}$ is the solution vector,

$$\|x - x^{(0)}\| = \sum_{j=1}^n |x_j - x_j^{(0)}|$$

Suppose each component of x mutates with a probability p . Let $I(p)$ be the probability that the resulting mutant constitutes an improvement over its "parent." For what mutation probability p does $I(p)$ assume a maximum? This question can be answered easily.

Suppose ζ of the n components of x disagree with $x^{(0)}$ while the others agree. A mutation among the components that agree will produce a degeneration while a mutation among the ζ "bad" components leads to an improvement.

The probability for at least one improvement in the "bad" part and no degeneration in the "good part" is:

$$I_0(p) = (1 - (1 - p)\zeta)(1 - p)^{n-\zeta} = (1 - p)^{n-\zeta} - (1 - p)^n$$

$I(p)$ is composed of this part plus $I_1(p) + I_2(p) + \dots + I_{\zeta-1}(p)$, the probability of having exactly one degeneration combined with at least two improvements, etc.,

$$I_1(p) = \left[1 - (1 - (1 - p)^2)^{(n-\zeta)(n-\zeta-1)/2} \right] \left[1 - (1 - p^2)^{\frac{\zeta-1}{2}} \right],$$

etc. developing $I_0(p)$ into a binomial series we find $I_0(p) = \zeta p + \zeta \left(\frac{\zeta-2n+1}{2} \right) p^2 + \dots$. Developing $I_1(p)$ we observe that the first factor is less than 1 while the second gives $\frac{(\zeta-1)}{2} p^2 + \dots$.

We have $\frac{dI_0(p)}{dp} = (n - \zeta)(1 - p)^{n-\zeta-1} - n(1 - p)^{n-1} = 0$. Solving for p

$$\text{we have } 1 - \frac{\zeta}{n} = (1 - p)\zeta \text{ and } p = 1 - \sqrt[1 - \frac{\zeta}{n}]{\frac{\zeta}{n}}$$

Assuming that ζ is small compared with n we have $p \approx \frac{1}{n}$.

For this value, however, the contribution of $I_2(p)$ is small as compared with $I_0(p)$. The development of $I_3(p)$ begins with p^3 , etc. Neglecting these contributions we find: If ζ is small as compared with n , then $I(p)$ takes its maximum for $p = \frac{1}{n}$, that means that in the average, one gene per individual per generation is mutated.

In *Drosophila*, surprisingly enough this condition is nearly satisfied. The estimated number of genes is 10^4 . The observed mutation rate (W. S. Stone²⁷) per locus is 10^{-4} to 10^{-5} . Taking into consideration that some genes have more than two alleles, this constitutes excellent agreement.

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