PART V. APPENDIX: OPTIMAL MUTATION PROBABILITY

Suppose we have a fitness function that is a function of \( n \) binary variables and is a monotone function of the Hamming distance of its argument vector from the solution vector: \( f(x) = g(||x - x^{(o)}||) \), where \( g \) is a monotone function defined on the integers \( 0, \ldots, n \), \( x^{(o)} \) is the solution vector,
\[
||x - x^{(o)}|| = \sum_{j=1}^{n} |x_j - x_j^{(o)}|
\]

Suppose each component of \( x \) mutates with a probability \( p \). Let \( I(p) \) be the probability that the resulting mutant constitutes an improvement over its "parent." For what mutation probability \( p \) does \( I(p) \) assume a maximum? This question can be answered easily.

Suppose \( \zeta \) of the \( n \) components of \( x \) disagree with \( x^{(o)} \) while the others agree. A mutation among the components that agree will produce a degeneration while a mutation among the \( \zeta \) "bad" components leads to an improvement.

The probability for at least one improvement in the "bad" part and no degeneration in the "good part" is:

\[
I_\zeta(p) = (1 - (1 - p) \zeta) (1 - p)^{n-\zeta} = (1 - p)^{n-\zeta} - (1 - p)^n
\]

\( I(p) \) is composed of this part plus \( I_1(p) + I_2(p) + \ldots + I_{n-1}(p) \), the probability of having exactly one degeneration combined with at least two improvements, etc.,

\[
I_1(p) = \left[1 - (1 - (1-p)^2)^{(n-\zeta)(n-\zeta+1)/2}\right] \left[1 - (1-p)^{\zeta-1}\right]
\]

etc. developing \( I_\zeta(p) \) into a binomial series we find \( I_\zeta(p) = \zeta p + \zeta \left(\frac{\zeta-2n+1}{2}\right) p^3 + \ldots \). Developing \( I_1(p) \) we observe that the first factor is less than 1 while the second gives \( \frac{(\zeta-1)}{2} p^3 + \ldots \).

We have
\[
\frac{dI_\zeta(p)}{dp} = (n-\zeta) (1-p)^{n-\zeta-1} - n(1-p)^{n-1} = 0.
\]
Solving for \( p \)

we have
\[
1 - \frac{\zeta}{n} = (1 - p) \zeta \quad \text{and} \quad p = 1 - \frac{\zeta}{n} \sqrt{1 - \frac{\zeta}{n}}
\]

Assuming that \( \zeta \) is small compared with \( n \) we have \( p \approx \frac{1}{n} \).
For this value, however, the contribution of $I_2(p)$ is small as compared with $I_3(p)$. The development of $I_3(p)$ begins with $p^3$, etc. Neglecting these contributions we find: If $\zeta$ is small as compared with $n$, then $I(p)$ takes its maximum for $p = \frac{1}{n}$, that means that in the average, one gene per individual per generation is mutated.

In Drosophila, surprisingly enough this condition is nearly satisfied. The estimated number of genes is $10^4$. The observed mutation rate (W. S. Stone) per locus is $10^{-4}$ to $10^{-5}$. Taking into consideration that some genes have more than two alleles, this constitutes excellent agreement.

REFERENCES